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Nowhere differentiable functions of analytic type on products of finitely connected planar domains

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Abstract Using Laurent decomposition and Mergelyan's theorem combined with Baire's category theorem, we prove generic nowhere differentiability on the distinguished boundary of functions of analytic type on products of planar domains bounded by finitely many disjoint Jordan curves. The parametrization of the boundaries are those induced by the natural Riemann maps.

Keywords Nowhere differentiable functions · Baire's theorem · Riemann mapping · Rational approximation · Several complex variables · Laurent decomposition

Mathematics Subject Classification Primary 26A27; Secondary 30H50

1 Introduction

In the nineteenth century, it was widely believed that every continuous function defined on \mathbb{R} has derivative at a significant set of points. Weierstrass was the first one to

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disprove this claim by providing an explicit example of a function $u : \mathbb{R} \rightarrow \mathbb{R}$ which is continuous, 2π -periodic, but not differentiable at any real number. Since then the existence of such functions attracted the interest of many mathematicians.

In fact, as Banach and Mazurkiewicz independently proved, in [2] and [9] respectively, that a topologically generic continuous function on a compact interval J of \mathbb{R} is nowhere differentiable. In [3], an analogue of this theorem has been proved for functions in the disc algebra $A(D)$, i.e. continuous functions defined on the closed unit disc \overline{D} , which are also holomorphic on D . In particular, it is proved that generically for every function $f \in A(D)$ both functions $\operatorname{Re} f|_{\mathbb{T}}$ and $\operatorname{Im} f|_{\mathbb{T}}$ are nowhere differentiable. The result is even extended for functions $f : \overline{D}^I \rightarrow \mathbb{C}$ that belong to the algebra $A(D^I)$, namely f is continuous on \overline{D}^I , which is endowed with the cartesian topology, and separately holomorphic on D^I .

In Sect. 4, we prove a similar result in the case of a domain Ω bounded by a finite number of disjoint Jordan curves. If $\Omega_0, \Omega_1, \dots, \Omega_n$ are the complements of the connected components of $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$, then the Riemann mappings $\phi_i : \overline{D} \rightarrow \overline{\Omega}_i$, $i = 1, \dots, n$ yield a parametrization of $\partial\Omega_i$, $i = 0, 1, \dots, n$, with parameter $\theta \in \mathbb{R}$. We will prove that generically for every $f \in A(\Omega)$ the functions $\operatorname{Re}(f \circ \phi_i)|_{\mathbb{T}}$ and $\operatorname{Im}(f \circ \phi_i)|_{\mathbb{T}}$ are nowhere differentiable with respect to $\theta \in \mathbb{R}$. In order to do so, we combine the Laurent decomposition of every function $f \in A(\Omega)$ with the results of the previous sections which deal with the case of the disc or more generally with the case of Jordan domains.

In Sect. 5, we generalize the above result in the context of several complex variables. Specifically, if we consider sets Ω which are the finite or countably infinite cartesian products of domains in \mathbb{C} as in Sect. 4, then a parametrization of the distinguished boundary of Ω arises from the Riemann mappings of the factors Ω_i of the product Ω . We prove that, generically, for every $f \in A(\Omega)$ neither the real part $\operatorname{Re} f$, nor the imaginary part $\operatorname{Im} f$ have directional derivatives at any point of the distinguished boundary of $\Omega = \prod_{i \in I} \Omega_i$ with respect to the above parametrization, for any direction $v \in \ell^\infty \setminus \{0\}$. We first treat the case of finite products that is when the set I is finite. In the case of infinitely many variables, we use the result for the case of finitely many variables and the fact that every $f \in A(\Omega)$ can be uniformly approximated by functions depending on a finite set of variables ([4, 8]).

2 The open unit disk

We denote the open unit disk in \mathbb{C} by D and $\mathbb{T} = \partial D$. Let Ω be a domain in \mathbb{C} . Although the only domains we will deal with in this sections are D and $\hat{\mathbb{C}} \setminus \overline{D}$ we will give a general definition which will be useful later.

Definition 2.1 Let Ω be a domain in \mathbb{C} . A function $f : \overline{\Omega} \rightarrow \mathbb{C}$ belongs to $A(\Omega)$ if it is continuous on $\overline{\Omega}$ and holomorphic on Ω . We endow $A(\Omega)$ with the topology of uniform convergence on compact subsets of $\overline{\Omega}$, where the closure of Ω is taken in \mathbb{C} .

We note that a function defined on \mathbb{T} can equivalently be thought as a 2π -periodic function defined on \mathbb{R} . Thus, by abuse of notation we write $u(y)$ instead of $u(e^{iy})$ for $y \in \mathbb{R}$.

Definition 2.2 A function $f \in C(\mathbb{T})$ belongs to Z if for every $\theta \in \mathbb{R}$

$$\limsup_{y \rightarrow \theta^+} \left| \frac{\operatorname{Re} f(y) - \operatorname{Re} f(\theta)}{y - \theta} \right| = +\infty \quad (2.1)$$

and

$$\limsup_{y \rightarrow \theta^+} \left| \frac{\operatorname{Im} f(y) - \operatorname{Im} f(\theta)}{y - \theta} \right| = +\infty. \quad (2.2)$$

In [3], it has been proven that the set of functions $f \in A(D)$ such that $f|_{\mathbb{T}}$ belongs to Z , which we denote by $Z(D)$, is a dense and G_δ subset of the disk algebra $A(D)$.

It will be useful to restrict ourselves to functions $f \in A(D)$, such that $f(0) = 0$. We denote the space of such functions by $A_0(D)$ and we endow it with the subspace topology. Using the above result we are going to prove that the functions $f \in A_0(D)$ such that $f|_{\mathbb{T}} \in Z$ is dense and G_δ in $A_0(D)$. Denote this set by $Z_0(D)$.

Proposition 2.3 *The set of functions $f \in A_0(D)$ such that $f|_{\mathbb{T}}$ belongs to Z is a dense and G_δ subset of $A_0(D)$.*

Proof Observe that $Z_0(D) = Z(D) \cap A_0(D)$. Consider the mapping

$$\begin{aligned} L : A_0(D) &\longrightarrow A(D) \\ f &\longmapsto f. \end{aligned}$$

L is obviously continuous and it is easy to see that $L^{-1}(Z(D)) = Z_0(D)$ is G_δ . So it suffices to show that Z_0 is dense in $A_0(D)$.

Let $\epsilon > 0$ and $f \in A_0(D) \subset A(D)$. Since $Z(D)$ is dense in $A(D)$ there exists a $g \in Z(D)$ such that

$$\|f - g\|_\infty < \frac{\epsilon}{2}. \quad (2.3)$$

More specifically

$$|f(0) - g(0)| < \frac{\epsilon}{2}. \quad (2.4)$$

It is now obvious that $\tilde{g}(z) = g(z) - g(0) \in A_0(D)$ and $\tilde{g} \in Z_0(D)$. Using the triangular inequality we deduce that

$$\|f - \tilde{g}\|_\infty < \epsilon, \quad (2.5)$$

which completes the proof. \square

We will now prove similar results for a subspace of $A(\hat{\mathbb{C}} \setminus \overline{D})$ denoted by $A_0(\hat{\mathbb{C}} \setminus \overline{D})$. In general, if the domain Ω is the complement of a compact set, then a function f belongs to $A_0(\Omega)$ if $f \in A(\Omega)$ and

$$\lim_{z \rightarrow \infty} f(z) = 0. \quad (2.6)$$

Proposition 2.4 *The set of functions $f \in A_0(\hat{\mathbb{C}} \setminus \overline{D})$ such that $f|_{\mathbb{T} \in Z}$ is a dense and G_δ subset of $A_0(\hat{\mathbb{C}} \setminus \overline{D})$.*

Proof Consider the mappings

$$\begin{aligned}\phi : \overline{D} &\longrightarrow \hat{\mathbb{C}} \setminus D \\ z &\longmapsto \frac{1}{z}\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}l : A_0(\hat{\mathbb{C}} \setminus \overline{D}) &\longrightarrow A_0(D) \\ f &\longmapsto f \circ \phi.\end{aligned}\tag{2.8}$$

We can easily see that l is continuous, onto $A_0(D)$ and

$$\|l(f) - l(g)\|_\infty = \|f - g\|_\infty.\tag{2.9}$$

Hence, l is an isometric isomorphism, which implies that the set $Z_0(\hat{\mathbb{C}} \setminus \overline{D}) := l^{-1}(Z_0(D))$ is dense and G_δ in $A_0(\hat{\mathbb{C}} \setminus \overline{D})$.

We will now prove that $Z_0(\hat{\mathbb{C}} \setminus \overline{D})$ coincides with the set of functions $f \in A_0(\hat{\mathbb{C}} \setminus \overline{D})$ such that $f|_{\mathbb{T} \in Z}$. This can be derived from the following equalities:

$$\left| \frac{u(y) - u(\theta)}{y - \theta} \right| = \left| \frac{(u \circ \phi)(-y) - (u \circ \phi)(-\theta)}{(-y) - (-\theta)} \right|,\tag{2.10}$$

$$\left| \frac{v(y) - v(\theta)}{y - \theta} \right| = \left| \frac{(v \circ \phi)(-y) - (v \circ \phi)(-\theta)}{(-y) - (-\theta)} \right|,\tag{2.11}$$

where u, v are the real and imaginary part of a function $f \in A_0(D)$ respectively and $\theta, y \in \mathbb{R}$ are such that $y \neq \theta$. \square

3 Jordan domains

In this section we replace the unit circle \mathbb{T} by a Jordan curve J and we prove similar results for functions defined on the Jordan domain Ω or on $\hat{\mathbb{C}} \setminus \overline{\Omega}$. The Jordan curve theorem reassures that the complement of a Jordan curve in \mathbb{C} has exactly two connected components, one bounded and one unbounded. We define a Jordan domain to be the bounded connected component of the complement of a Jordan curve on \mathbb{C} . For such domains Ω , every Riemann mapping $\phi : D \rightarrow \Omega$ has a continuous and injective extension over the closed the unit disk \overline{D} ([7]). We claim that generically for every function $f \in A(\Omega)$, the function $(f \circ \phi)|_{\mathbb{T}}$ belongs to Z . In what follows, we prove the aforementioned assertions.

Theorem 3.1 *Let Ω be a Jordan domain and $\phi : \overline{D} \rightarrow \overline{\Omega}$ a Riemann mapping. The set of functions $f \in A(\Omega)$ such that $(f \circ \phi)|_{\mathbb{T}}$ belongs to Z is a dense and G_δ subset of $A(\Omega)$.*

Proof Let

$$\begin{aligned} L : A(\Omega) &\longrightarrow A(D) \\ f &\longmapsto f \circ \phi. \end{aligned} \quad (3.1)$$

It is easy to verify that L is an isometric isomorphism, which implies that the set $L^{-1}(Z(D))$ is a dense and G_δ subset of $A(\Omega)$.

Furthermore $L^{-1}(Z(D))$ coincides with the set of functions $f \in A(\Omega)$ such that $(f \circ \phi)|_{\mathbb{T}}$ belongs to Z , because, for any $f \in A(\Omega)$ we have that: $(f \circ \phi)|_{\mathbb{T}} \in Z \Leftrightarrow L(f)|_{\mathbb{T}} \in Z \Leftrightarrow L(f) \in Z(D) \Leftrightarrow f \in L^{-1}(Z(D))$. \square

Theorem 3.2 *Let J be a Jordan curve and Ω the unbounded connected component of $\mathbb{C} \setminus J^*$. Let also $\phi : \overline{D} \rightarrow \overline{\Omega} \cup \{\infty\}$ be a Riemann mapping. The set of functions $f \in A(\Omega \cup \{\infty\})$ such that $(f \circ \phi)|_{\mathbb{T}}$ belongs to Z is a dense and G_δ subset of $A_0(\Omega \cup \{\infty\})$.*

Proof Consider the mapping

$$\begin{aligned} \mathcal{L} : A_0(\Omega \cup \{\infty\}) &\longrightarrow A_0(D) \\ f &\longmapsto f \circ \phi. \end{aligned} \quad (3.2)$$

Similarly to Proposition 2.4 the desired set is a dense and G_δ subset of $A_0(\Omega \cup \{\infty\})$. \square

Remark 3.3 In Theorems 3.1 and 3.2 every Riemann mapping $\Phi : D \rightarrow \Omega$ can be obtained from the composition of a fixed Riemann mapping ϕ of Ω with a Möbius transformation mapping D onto itself. Therefore, one can easily see that if the functions $\operatorname{Re} (f \circ \phi)|_{\mathbb{T}}$ and $\operatorname{Im} (f \circ \phi)|_{\mathbb{T}}$ are nowhere differentiable with respect to $\theta \in \mathbb{R}$, then the same holds true for the functions $\operatorname{Re} (f \circ \Phi)|_{\mathbb{T}}$ and $\operatorname{Im} (f \circ \Phi)|_{\mathbb{T}}$.

The statement of Remark 3.3 does not hold in general if we replace Φ by an arbitrary parametrization. In what follows we prove that if a function $f_0 \in A(D)$ is nowhere differentiable with respect to the natural parametrization and $\operatorname{Re} f_0|_{\mathbb{T}}$, $\operatorname{Im} f_0|_{\mathbb{T}}$ are a -Hölder continuous, then there is a parametrization s of \mathbb{T} such that $f_0|_{\mathbb{T}}$ is differentiable at some point with respect to s .

Suppose that f_0 is a function as above. Let $s(t) = t^b$ for some $b > 1/a$. It is easy to see that $\operatorname{Re} (f_0 \circ s)|_{\mathbb{T}}$ and $\operatorname{Im} (f_0 \circ s)|_{\mathbb{T}}$ are differentiable at 0 with respect to s and their derivatives are equal to 0. Indeed, this follows from the fact that

$$\frac{|u_0(s(t)) - u_0(s(0))|}{|t|} \leq c \frac{|t|^{ab}}{|t|}$$

for some constant $c > 0$, where $u_0 = \operatorname{Re} f_0$. The last expression converges to 0 as $t \rightarrow 0$, which implies that the derivative of $(u_0 \circ s)|_{\mathbb{T}}$ at 0 is equal to 0. Similarly, the derivative of $\operatorname{Im} (f_0 \circ s)|_{\mathbb{T}}$ at 0 is 0.

To give an example of such a function f_0 consider the Weierstrass function $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$u_0(x) = \sum_{n=0}^{\infty} 2^{-an} \cos(2^n x)$$

for $0 < a < 1$. Since u_0 is 2π -periodic, we can regard u_0 as a function defined on \mathbb{T} . Note that $u_0 = \operatorname{Re}(f_0)$ where $f_0(z) = \sum_{n=0}^{\infty} 2^{-an} z^{2^n}$. The latter function belongs to $A(D)$ as the M-test shows. A classic result of G. H. Hardy [6] states that both u_0 and its harmonic conjugate \tilde{u}_0 are nowhere differentiable and a -Hölder continuous.

On the other hand, a nowhere differentiable function with respect to a parametrization s_1 of a Jordan curve J is also nowhere differentiable with respect to another parametrization s_2 whenever they satisfy $s_2 = s_1 \circ r$ for a differentiable function $r : \mathbb{T} \rightarrow \mathbb{T}$ with $r'(t) \neq 0$ for every t . Indeed, consider a function $f \in A(\Omega)$ such that $(f \circ s_1)|_{\mathbb{T}} \in Z$. Then we have that

$$\frac{f(s_2(t)) - f(s_2(t_0))}{t - t_0} = \frac{f(s_1(r(t))) - f(s_1(r(t_0)))}{r(t) - r(t_0)} \frac{r(t) - r(t_0)}{t - t_0}$$

which implies that $(f \circ s_2)|_{\mathbb{T}} \in Z$. Hence, a nowhere differentiable function with respect to some parametrization is nowhere differentiable with respect to every other “nice” reparametrization.

4 Domains bounded by a finite number of disjoint Jordan curves

Now we consider a more general setting, where Ω is a bounded domain whose boundary consists of a finite number of disjoint Jordan curves. If V_0, \dots, V_{n-1} are the connected components of $\hat{\mathbb{C}} \setminus \Omega$, $\infty \in V_0$ and $\Omega_0 = \hat{\mathbb{C}} \setminus V_0, \dots, \Omega_{n-1} = \hat{\mathbb{C}} \setminus V_{n-1}$, then there exist Riemann mappings $\phi_i : \bar{D} \rightarrow \bar{\Omega}_i$. We are interested in functions $f \in A(\Omega)$, such that $(f \circ \phi_i)|_{\mathbb{T}}$ belongs to Z , for all $i = 0, \dots, n-1$.

The main result of this section is that the aforementioned class of functions is G_δ dense in $A(\Omega)$.

Theorem 4.1 *The class of functions $f \in A(\Omega)$ such that for every $i \in \{0, 1, \dots, n-1\}$, the functions $f \circ \phi_i \in Z$ is dense and G_δ in $A(\Omega)$.*

We will prove the theorem using the results of the previous sections, the Laurent decomposition and Baire category theorem.

At first let us give some necessary definitions. For every $k \in \mathbb{N}$ and $i \in \{0, 1, \dots, n-1\}$ consider the sets

$$D_k = \{u \in C_{\mathbb{R}}(T) : \text{for every } \theta \in \mathbb{R} \text{ there exists } y \in \left(\theta, \theta + \frac{1}{k}\right) \text{ such that } |u(y) - u(\theta)| > k|y - \theta|\}$$
(4.1)

and

$$E_k^{(i)} = \{f \in A(\Omega) : \operatorname{Re}(f \circ \phi_i) \upharpoonright_{\mathbb{T}} \in D_k\}. \quad (4.2)$$

Lemma 4.2 *For every $k \in \mathbb{N}$ and $i \in \{0, 1, \dots, n-1\}$ the set $E_k^{(i)}$ is an open subset of $A(\Omega)$.*

Proof Let $k \in \mathbb{N}$ and $i \in \{0, 1, \dots, n-1\}$. We will prove equivalently that $A(\Omega) \setminus E_k^{(i)}$ is a closed subset of $A(\Omega)$. Let f_m be a sequence in $A(\Omega) \setminus E_k^{(i)}$ converging uniformly on $\overline{\Omega}$ to a function $f \in A(\Omega)$. For every $m \in \mathbb{N}$ there exist $\theta_m \in \mathbb{R}$ such that

$$\left| \frac{u_m(\phi_i(y)) - u_m(\phi_i(\theta_m))}{y - \theta_m} \right| \leq k, \quad (4.3)$$

where $u_m = \operatorname{Re} f_m$. Since every u_m is 2π -periodic we may assume that every $\theta_m \in [0, 2\pi]$ and thus the sequence $\{\theta_m\}$ has a convergent subsequence. Without loss of generality we assume that $\{\theta_m\}$ converges to a $\theta \in [0, 2\pi]$.

Let $y \in \left(\theta, \theta + \frac{1}{k}\right)$. We can easily deduce that there exists $m_0 \in \mathbb{N}$, such that for every $m \geq m_0$ it is true that $y \in \left(\theta_m, \theta_m + \frac{1}{k}\right)$. Thus, applying (4.3) and using the convergence of $\{\theta_m\}$ to θ and the uniform convergence of $\{f_m\}$ to f we have that

$$\left| \frac{u(\phi_i(y)) - u(\phi_i(\theta))}{y - \theta} \right| \leq k, \quad (4.4)$$

where $u = \operatorname{Re} f$. Hence (4.4) holds for every $y \in \left(\theta, \theta + \frac{1}{k}\right)$ and thus $f \in A(\Omega) \setminus E_k^{(i)}$. Therefore, $A(\Omega) \setminus E_k^{(i)}$ is a closed subset of $A(\Omega)$. \square

Theorem 4.3 *The class*

$$S = \bigcap_{i=0}^{n-1} \bigcap_{k=1}^{\infty} E_k^{(i)} \quad (4.5)$$

is a dense and G_δ subset of $A(\Omega)$.

Proof Lemma 4.2 indicates that for every $i \in \{0, 1, \dots, n-1\}$, the sets

$$S_i = \bigcap_{k=1}^{\infty} E_k^{(i)} \quad (4.6)$$

are G_δ . We will prove that S_0 is dense in $A(\Omega)$.

Let $f \in A(\Omega)$ and $f_i \in A(\Omega_i)$ be such that $f = f_0 + f_1 + \dots + f_{n-1}$. Such a decomposition is possible because f is analytic on Ω , $\hat{\mathbb{C}} \setminus \Omega$ has n connected components, therefore we can apply the global Cauchy theorem on $\frac{f(\zeta) - f(z)}{\zeta - z}$, each time, integrating on appropriate curves that depend on $z \in \Omega$ [1]. Let also $g =$

$f_1 + \dots + f_{n-1}$. Since $\overline{\Omega} \subset \overline{\Omega_0}$ is compact, $\hat{\mathbb{C}} \setminus \Omega$ has n connected components and $\phi_0^{-1} : \overline{\Omega_0} \rightarrow \mathbb{C}$ is a conformal map, the set $\phi_0^{-1}(\overline{\Omega}) \subset \overline{D} \subset \mathbb{C}$ is compact and $\hat{\mathbb{C}} \setminus \phi_0^{-1}(\overline{\Omega})$ has n connected components. The function $g \circ \phi_0 : \phi_0^{-1}(\overline{\Omega}) \rightarrow \mathbb{C}$ belongs to $A(\phi_0^{-1}(\overline{\Omega}))$ and by Mergelyan's theorem there exists a sequence $\{r_m\}$ of rational functions with poles off $\phi_0^{-1}(\overline{\Omega})$ which converges to $g \circ \phi_0$ uniformly on $\phi_0^{-1}(\overline{\Omega})$. Thus, the sequence $\{r_m \circ \phi_0^{-1}\}$ converges to g uniformly on $\overline{\Omega}$.

Since $f_0 \in A(\Omega_0)$, by Theorem 3.1 there exists a sequence $\{g_m\}$, such that $g_m \in A(\Omega_0)$ and $g_m \circ \phi_0 \in Z(D)$, which converges to f_0 . From the above we have that the sequence $\{g_m + r_m \circ \phi_0^{-1}\}$ converges uniformly to f in $A(\Omega)$ and also that:

$$\begin{aligned} & \left| \frac{\operatorname{Re}(g_m + r_m \circ \phi_0^{-1})(\phi_0(y)) - \operatorname{Re}(g_m + r_m \circ \phi_0^{-1})(\phi_0(\theta))}{y - \theta} \right| \\ & \geq \left| \frac{\operatorname{Re} g_m(\phi_0(y)) - \operatorname{Re} g_m(\phi_0(\theta))}{y - \theta} \right| \\ & \quad - \left| \frac{\operatorname{Re}(r_m \circ \phi_0^{-1})(\phi_0(y)) - \operatorname{Re}(r_m \circ \phi_0^{-1})(\phi_0(\theta))}{y - \theta} \right| \\ & = \left| \frac{\operatorname{Re} g_m(\phi_0(y)) - \operatorname{Re} g_m(\phi_0(\theta))}{y - \theta} \right| - \left| \frac{\operatorname{Re} r_m(y) - \operatorname{Re} r_m(\theta)}{y - \theta} \right| \end{aligned} \quad (4.7)$$

for every $\theta, y \in \mathbb{R}, \theta \neq y$. From (4.7) and the fact that the functions r_m are differentiable at the unit circle, every $g_m + r_m \circ \phi_0^{-1}$ belongs to S_0 . Therefore, the set S_0 is dense.

Similarly, the sets S_1, S_2, \dots, S_{n-1} are dense and Baire's theorem completes the proof. \square

Proof of Theorem 4.1 It is easy to see that $S \cap iS$ is a subset of all functions $f \in A(\Omega)$ such that for every $i \in \{0, 1, \dots, n-1\}$ the functions $(\operatorname{Re} f \circ \phi_i)|_{\mathbb{T}}$ and $(\operatorname{Im} f \circ \phi_i)|_{\mathbb{T}}$ have no derivatives at any point. Since S is G_δ and dense in $A(\Omega)$ and multiplication by i is an automorphism of $A(\Omega)$, we conclude that iS is G_δ and dense in $A(\Omega)$. Applying Baire's category theorem we have that $S \cap iS$ is G_δ dense in $A(\Omega)$. \square

It is natural to ask if our results can be generalised to domains bounded by an infinite number of Jordan curves.

Question 4.4 *Can our results be generalised to domains Ω bounded by an infinite number of Jordan curves?*

5 The case of several variables

Let I be a non empty finite or countably infinite set and $\{\Omega_i\}_{i \in I}$ be a family of domains in \mathbb{C} , all of them bounded by a finite number of disjoint Jordan curves as in Sect. 4. We denote by k_i the degree of connectivity of Ω_i . In accordance to the notation of the previous sections, we denote by $V_{i,j}, i \in I, j \in \{0, \dots, k_i - 1\}$ the connected

components of $\hat{\mathbb{C}} \setminus \overline{\Omega_i}$ and $\Omega_{i,j} = \hat{\mathbb{C}} \setminus \overline{V_{i,j}}$. Let also $\infty \in \Omega_{i,0}$ and $\phi_{i,j} : \overline{D} \rightarrow \overline{\Omega_{i,j}}$ be some fixed Riemann maps.

We consider the spaces $\Omega = \prod_{i \in I} \Omega_i$, $\overline{\Omega} = \prod_{i \in I} \overline{\Omega_i}$ and $b_0\Omega = \prod_{i \in I} \partial\Omega_i$, the distinguished boundary of Ω , all of them endowed with the product topology. According to [4, 5] and [8] we give the following definition.

Definition 5.1 A function $f : \overline{\Omega} \rightarrow \mathbb{C}$ belongs to the space $A(\Omega)$ if it is continuous on $\overline{\Omega}$, endowed with the product topology, and separately holomorphic on Ω . By separately holomorphic we mean that if all coordinates but one are fixed, say the coordinate i_0 , then the restriction of f is a holomorphic function of the variable z_{i_0} in Ω_{i_0} . We endow the space $A(\Omega)$ with the supremum norm.

We also consider the set

$$R = \prod_{i \in I} \{0, \dots, k_i - 1\} \quad (5.1)$$

and the functions

$$\begin{aligned} \phi_r : \overline{D}^I &\longrightarrow \overline{\Omega_{1,r_1}} \times \overline{\Omega_{2,r_2}} \times \dots = \prod_{i \in I} \overline{\Omega_{i,r_i}} \\ (z_i)_{i \in I} &\longmapsto (\phi_{i,r_i}(z_i))_{i \in I} \end{aligned} \quad (5.2)$$

where $r = (r_i)_{i \in I} \in R$.

We are going to prove that, generically, for every $f \in A(\Omega)$ the functions $\operatorname{Re}(f \circ \phi_r) \upharpoonright_{\mathbb{T}^I}$ and $\operatorname{Re}(f \circ \phi_r) \upharpoonright_{\mathbb{T}^I}$ have no directional derivatives for any $r \in R$ and any direction $v \in \ell^\infty(I) \setminus \{0\}$. In order to do so, we will first treat the case where I is a finite set and separately the case where I is a countably infinite set.

5.1 Finitely many variables

Let I be a non empty finite set.

We consider the sets

$$\begin{aligned} D_{n,k,s}(I) = \Big\{ u \in C_{\mathbb{R}}(\mathbb{T}^I) : &\text{for every } \theta \in \mathbb{R}^I \text{ and every direction} \\ &v \in \mathbb{R}^I \text{ with } \frac{1}{s} \leq |v_k| \leq s \text{ and } |v_j| \leq s \text{ for every } j \in I \text{ there exists} \\ &y \in \left(\theta - \frac{1}{n}v, \theta + \frac{1}{n}v \right) \text{ such that } |u(\theta) - u(y)| > n\|y - \theta\|_\infty \Big\} \end{aligned} \quad (5.3)$$

for some $n, s \in \mathbb{N}$ and $k \in I$. Also consider:

$$E_{n,k,s}^{(r)}(I) = \{f \in A(\Omega) : \operatorname{Re}(f \circ \phi_r) \upharpoonright_{\mathbb{T}^I} \in D_{n,k,s}\} \quad (5.4)$$

for some $r \in R$.

Lemma 5.2 For every $n, s \in \mathbb{N}$, $k \in I$ and $r \in R$ the set $E_{n,k,s}^{(r)}(I)$ is an open subset of $A(\Omega)$.

Proof We will prove that the set $A(\Omega) \setminus E_{n,k,s}^{(r)}(I)$ is closed. We consider a sequence $\{f_m\}$ such that every $f_m \in A(\Omega) \setminus E_{n,k,s}^{(r)}(I)$, which uniformly converges to a function $f \in A(\Omega)$. For each m there exists $\theta^{(m)} \in \mathbb{T}^I$ and a direction $v^{(m)} \in \mathbb{R}^I$ with $|v_j^{(m)}| \leq s$ for all $j \in I$ and $\frac{1}{s} \leq |v_k^{(m)}| \leq s$, such that:

$$|(u_m \circ \phi_r)(y) - (u_m \circ \phi_r)(\theta^{(m)})| \leq n \|y - \theta^{(m)}\|_\infty \quad (5.5)$$

for every $y \in \left(\theta^{(m)} - \frac{1}{n}v^{(m)}, \theta^{(m)} + \frac{1}{n}v^{(m)}\right)$, where $u_m = \operatorname{Re} f_m$. But \mathbb{T}^I and the set $\{v \in \mathbb{R}^I : |v_j| \leq s, \forall j \in I \text{ and } \frac{1}{s} \leq |v_k| \leq s\}$ are compact. Thus, the sequences $\{\theta^{(m)}\}_{m \in \mathbb{N}}$ and $\{v^{(m)}\}_{m \in \mathbb{N}}$ have converging subsequences. Without loss of generality we assume that $\theta^{(m)} \rightarrow \theta \in \mathbb{T}^I$ and $v^{(m)} \rightarrow v \in \mathbb{R}^I$ with $|v_j| \leq s$ for all $j \in I$ and $\frac{1}{s} \leq |v_k| \leq s$.

If $y \in \left(\theta - \frac{1}{n}v, \theta + \frac{1}{n}v\right)$, then there exists $-\frac{1}{n} < t < \frac{1}{n}$ such that

$$y = \theta + tv = \lim_{m \rightarrow \infty} (\theta^{(m)} + tv^{(m)}). \quad (5.6)$$

Since f_m converges uniformly to f , it follows that u_m converge uniformly to u , where $u = \operatorname{Re} f$ and ϕ_r is continuous which implies that $\phi_r(\theta^{(m)} + tv^{(m)}) \rightarrow \phi_r(\theta + tv)$, as $m \rightarrow \infty$. The above combined with (5.5) imply that

$$|(u \circ \phi_r)(y) - (u \circ \phi_r)(\theta)| \leq n \|y - \theta\|_\infty \quad (5.7)$$

and therefore $f \in A(\Omega) \setminus E_{n,k,s}^{(r)}(I)$ which proves the lemma. \square

Now let

$$S_{k,s}^{(r)}(I) = \bigcap_{n=1}^{\infty} E_{n,k,s}^{(r)}(I) \quad (5.8)$$

for all $r \in R$, $k \in I$ and $s \in \mathbb{N}$. Obviously, $S_{k,s}^{(r)}(I)$ is G_δ in $A(\Omega)$.

Lemma 5.3 For all $r \in R$ and $k, s \in I$ the class $S_{k,s}^{(r)}(I)$ is dense in $A(\Omega)$.

Proof Let us show that $S_{k,s}^{(r)}(I)$ is non empty. We have proved in a previous section that there is a function $g \in A(\Omega_k)$ such that

$$\limsup_{y \rightarrow \theta^+} \left| \frac{\operatorname{Re}(g \circ \phi_{k,r_k})(y) - \operatorname{Re}(g \circ \phi_{k,r_k})(\theta)}{y - \theta} \right| = +\infty \quad (5.9)$$

for all $\theta \in \mathbb{R}$. The above implies that for all $\theta \in \mathbb{R}$:

$$\limsup_{t \rightarrow 0^+} \left| \frac{\operatorname{Re}(g \circ \phi_{k,r_k})(\theta + t\alpha) - \operatorname{Re}(g \circ \phi_{k,r_k})(\theta)}{t\alpha} \right| = +\infty \quad (5.10)$$

for any $\alpha \in \mathbb{R}$ with $\frac{1}{s} \leq |\alpha| \leq s$.

Now consider the function $f : \overline{\Omega} \rightarrow \mathbb{C}$ which maps $(z_i)_{i \in I}$ to $g(z_k)$. We will prove that $f \in S_{k,s}^{(r)}(I)$. Let $\theta \in \mathbb{R}^I$ and $v \in \mathbb{R}^I$ such that $|v_j| \leq s$ for all $j \in I$ and $\frac{1}{s} \leq |v_k| \leq s$. We observe that

$$\left| \frac{u_r(\theta + tv) - u_r(\theta)}{t \|v\|_\infty} \right| = \frac{|v_k|}{\|v\|_\infty} \left| \frac{\tilde{u}_{k,r_k}(\theta_k + tv_k) - \tilde{u}_{k,r_k}(\theta_k)}{tv_k} \right|, \quad (5.11)$$

does not remain bounded as $t \rightarrow 0$, where $u_r = \operatorname{Re} f \circ \phi_r$ and $\tilde{u}_{k,r_k} = \operatorname{Re} f \circ \phi_{k,r_k}$. Thus $f \in S_{k,s}^{(r)}(I)$ and $S_{k,s}^{(r)}(I) \neq \emptyset$.

Let us now show that $S_{k,s}^{(r)}(I)$ is dense in $A(\Omega)$. We remind the reader that

$$\begin{aligned} \phi_r : \overline{D}^I &\longrightarrow \overline{\Omega_{1,r_1}} \times \overline{\Omega_{2,r_2}} \times \cdots = \prod_{i \in I} \overline{\Omega_{i,r_i}} \\ (z_i)_{i \in I} &\longmapsto (\phi_{i,r_i}(z_i))_{i \in I} \end{aligned} \quad (5.12)$$

and that the function

$$\begin{aligned} \phi_r^{-1} : \overline{\Omega_{1,r_1}} \times \overline{\Omega_{2,r_2}} \times \cdots &\longrightarrow \overline{D}^I \\ (w_i)_{i \in I} &\longmapsto (\phi_{i,r_i}^{-1}(w_i))_{i \in I} \end{aligned} \quad (5.13)$$

is the inverse of ϕ_r . It easily follows that $\phi_r \in A(D^I)$ and $\phi_r^{-1} \in A(\Omega_{1,r_1} \times \Omega_{2,r_2} \times \cdots)$. Now, note that $\phi_r^{-1}(\overline{\Omega}) = \prod_{i \in I} \phi_{i,r_i}^{-1}(\overline{\Omega_i})$ where $\phi_{i,r_i}^{-1}(\overline{\Omega_i})$ is a compact set in \mathbb{C} of finite connectivity and consider the functions

$$\begin{aligned} \tilde{\phi}_r : \phi_r^{-1}(\overline{\Omega}) &\longrightarrow \overline{\Omega} \\ z &\longmapsto \phi_r(z) \end{aligned} \quad (5.14)$$

and

$$\begin{aligned} \widetilde{\phi_r^{-1}} : \overline{\Omega} &\longrightarrow \phi_r^{-1}(\overline{\Omega}) \\ z &\longmapsto \phi_r^{-1}(z). \end{aligned} \quad (5.15)$$

We observe that $\widetilde{\phi_r^{-1}}$ is the inverse of $\tilde{\phi}_r$. Also, note that if $h \in A(\Omega)$, then $h \circ \tilde{\phi}_r \in A(\phi_r^{-1}(\overline{\Omega}))$. Since $\phi_r^{-1}(\overline{\Omega})$ is a product of domains in \mathbb{C} bounded by a finite number

of disjoint Jordan curves, then according to the result of [4] in the case of finitely many variables, the set U of functions which are finite sums of finite products of rational functions of one complex variable z_i with poles off Ω_i is dense in $A(\phi_r^{-1}(\overline{\Omega}))$. Hence, the set $f + U \circ \widetilde{\phi_r^{-1}}$ is dense in $A(\Omega)$. If $q \in U$, then for every $t \in \mathbb{R}$, $\theta \in \mathbb{R}^I$ and $v \in \mathbb{R}^I$ with $|v_j| \leq s$ for all $j \in I$ and $\frac{1}{s} \leq |v_k| \leq s$ we have that:

$$\left| \frac{(q \circ \widetilde{\phi_r^{-1}})(\phi_r(\theta + tv)) - (q \circ \widetilde{\phi_r^{-1}})(\phi_r(\theta))}{t \|v\|_\infty} \right| = \frac{1}{\|v\|_\infty} \left| \frac{q(\theta + tv) - q(\theta)}{t} \right| \quad (5.16)$$

remains bounded. This follows from the fact that every rational function of one complex variable z_i which appears in q has poles off $\overline{\Omega_i}$. Thus, the function $p(t) = q(\theta + tv)$ is differentiable at 0. Therefore, $f + U \circ \widetilde{\phi_r^{-1}} \subset S_{k,s}^{(r)}(I)$, which completes the proof. \square

Baire's theorem combined with Lemmas 5.2 and 5.3 imply the following theorem.

Theorem 5.4 *The class*

$$S(I) = \bigcap_{r \in R} \bigcap_{k \in I} \bigcap_{s \in \mathbb{N}} S_{k,s}^{(r)}(I) \quad (5.17)$$

is a dense and G_δ subset of $A(\Omega)$.

Consequently, $S(I) \cap iS(I)$ is dense and G_δ in $A(\Omega)$ and it is a subset of the set of all functions $f \in A(\Omega)$ such that both $\operatorname{Re}(f \circ \phi_r) \upharpoonright_{\mathbb{T}^I}$ and $\operatorname{Im}(f \circ \phi_r) \upharpoonright_{\mathbb{T}^I}$ are nowhere differentiable for any $r \in R$ and any direction $v \in \ell^\infty(I) \setminus \{0\}$.

It is natural to ask if our results hold for domains bounded by a countably infinite number of disjoint Jordan curves.

5.2 Infinitely many variables

Let I be an countably infinite set.

Our goal is to generalize the result of the previous subsection to the case of infinite many variables, namely we intend to prove that the set of functions $f \in A(\Omega)$ such that for every $\theta \in \prod_{i \in I} \partial \Omega_i$ and direction $u \in \ell^\infty \setminus \{0\}$,

$$\limsup_{t \rightarrow 0} \frac{|\operatorname{Re}(f)(\theta + tv) - \operatorname{Re}(f)(\theta)|}{\|tv\|_\infty} = +\infty$$

and

$$\limsup_{t \rightarrow 0} \frac{|\operatorname{Im}(f)(\theta + tv) - \operatorname{Im}(f)(\theta)|}{\|tv\|_\infty} = +\infty$$

is dense and G_δ . For technical reasons we need to replace the ℓ^∞ -norm in the above definition by the metric

$$\rho((z_i)_{i \in \mathbb{N}}, (w_i)_{i \in \mathbb{N}}) = \sum_{i=1}^{\infty} \frac{|z_i - w_i|}{2^i}$$

which defines the pointwise convergence. However, the two aforementioned sets coincide, since the ℓ^∞ -norm and ρ are equivalent in every line of the form $\theta + \mathbb{R}v$. Hence, it suffices to prove that the latter set is dense and G_δ .

Let us start by defining the sets

$$\begin{aligned} D_{n,k,s}(I) = \{ & u \in C_{\mathbb{R}}(\mathbb{T}^I) : \text{for every } \theta \in \mathbb{R}^I \text{ and every direction} \\ & v \in \mathbb{R}^I \text{ with } |v_j| \leq s \text{ for any } j \in I \text{ and } \frac{1}{s} \leq |v_k| \leq s \text{ there exists} \\ & y \in \left(\theta - \frac{1}{n}v, \theta + \frac{1}{n}v \right) \text{ such that } |u(\theta) - u(y)| > n\rho(y, \theta) \} \end{aligned} \quad (5.18)$$

and

$$E_{n,k,s}(I) = \bigcap_{r \in R} \{f \in A(\Omega) : \text{Re}(f \circ \phi_r) \upharpoonright_{\mathbb{T}^I} \in D_{n,k,s}\}, \quad (5.19)$$

for $n, s \in \mathbb{N}$ and $k \in I$.

Lemma 5.5 *For every $n, s \in \mathbb{N}$ and $k \in I$ the set $E_{n,k,s}(I)$ is an open subset of $A(\Omega)$.*

Proof We will prove that the set $A(\Omega) \setminus E_{n,k,s}(I)$ is closed. We consider a sequence $\{f_m\}$ such that every $f_m \in A(\Omega) \setminus E_{n,k,s}(I)$, which converges to a function $f \in A(\Omega)$. For each m there exist $r^{(m)} \in R$, $\theta^{(m)} \in \mathbb{T}^{\mathbb{N}}$ and a direction $v^{(m)} \in \mathbb{R}^{\mathbb{N}}$ with $|v_j^{(m)}| \leq s$ for all $j \in I$ and $\frac{1}{s} \leq |v_k^{(m)}| \leq s$, such that

$$|(u_m \circ \phi_{r^{(m)}})(y) - (u_m \circ \phi_{r^{(m)}})(\theta^{(m)})| \leq n\rho(y, \theta^{(m)}) \quad (5.20)$$

for every $y \in \left(\theta^{(m)} - \frac{1}{n}v^{(m)}, \theta^{(m)} + \frac{1}{n}v^{(m)} \right)$, where $u_m = \text{Re } f_m$. But R , $\mathbb{T}^{\mathbb{N}}$ and the set $\{v \in \mathbb{R}^{\mathbb{N}} : |v_j| \leq s, \forall j \in I \text{ and } \frac{1}{s} \leq |v_k| \leq s\}$ are metrizable and compact by Tychonoff's theorem. Thus, they have converging subsequences. Without loss of generality we assume that $r^{(m)} \rightarrow r \in R$, $\theta^{(m)} \rightarrow \theta \in \mathbb{R}^{\mathbb{N}}$ and $v^{(m)} \rightarrow v$, where $v \in \mathbb{R}^{\mathbb{N}}$ and $|v_j| \leq s$ for all $j \in I$ and $\frac{1}{s} \leq |v_k| \leq s$.

If $y \in \left(\theta - \frac{1}{n}v, \theta + \frac{1}{n}v \right)$, then there exists a $t \in \mathbb{R}$ with $-\frac{1}{n} < t < \frac{1}{n}$ such that:

$$y = \theta + tv = \lim_{m \rightarrow \infty} (\theta^{(m)} + tv^{(m)}). \quad (5.21)$$

Let $i \in I$. Since $R = \prod_{i \in I} \{0, \dots, k_i - 1\}$ is a product of finite sets and $r^{(m)} \rightarrow r \in \mathbb{R}$, there exists $m_0 \in \mathbb{N}$, such that

$$\phi_{i,r_i^{(m)}} = \phi_{i,r_i}, \forall m \geq m_0. \quad (5.22)$$

Since ϕ_{i,r_i} is continuous, we have that

$$\phi_{i,r_i^{(m)}}(\theta_i^{(m)} + t v_i^{(m)}) \rightarrow \phi_{i,r_i}(y_i), \quad \phi_{i,r_i^{(m)}}(\theta_i^{(m)}) \rightarrow \phi_{i,r_i}(\theta_i). \quad (5.23)$$

Hence,

$$\phi_{r_{(m)}}(y^{(m)}) \rightarrow \phi_r(y), \quad \phi_{r_{(m)}}(\theta^{(m)}) \rightarrow \phi_r(\theta) \quad (5.24)$$

in the product topology. Therefore, letting $m \rightarrow \infty$ we see that the left hand side of (5.20) converges to $|(u \circ \phi_r)(y) - (u \circ \phi_r)(\theta)|$ where $u = \operatorname{Re} f$, since $f_m \rightarrow f$ uniformly, while the right hand side converges to $\rho(y, \theta)$. Hence,

$$|(u \circ \phi_r)(y) - (u \circ \phi_r)(\theta)| \leq n\rho(y, \theta), \quad (5.25)$$

which implies that $f \in A(\Omega) \setminus E_{n,k,s}(I)$ and that the set $A(\Omega) \setminus E_{n,k,s}(I)$ is closed. \square

Theorem 5.6 *The class*

$$S(I) = \bigcap_{k \in I} \bigcap_{s=1}^{\infty} \bigcap_{n=1}^{\infty} E_{n,k,s}(I) \quad (5.26)$$

is a dense and G_δ subset of $A(\Omega)$.

Proof Lemma 5.5 implies that $S(I)$ is a G_δ . Thus, we only have to prove that $S(I)$ is dense. Let $F \in A(\Omega)$. From [4] there exist a sequence of finite sets $\{I_n\}$, $I_n \subset I$, and a sequence $\{F_n\}$ in $A(\Omega)$, such that each F_n depends only on the variables z_i , $i \in I_n$ and $F_n \rightarrow F$ uniformly. Let f_n be the restriction of F_n to $\prod_{i \in I_n} \Omega_i$. From Theorem 5.4 there exists a sequence $g_n \in S(I_n) = \bigcap_{r \in R(I_n)} \bigcap_{k \in I_n} \bigcap_{s=1}^{\infty} S_{k,s}^{(r)}(I_n)$, where $R(I_n) = \prod_{i \in I_n} \{0, \dots, k_i - 1\}$, such that $\|f_n - g_n\|_\infty \rightarrow 0$. It is easy to see that the functions $G_n : \overline{\Omega} \rightarrow \mathbb{C}$ which map $(z_i)_{i \in I}$ to $g_n((z_i)_{i \in I_n})$ respectively belong to $S(I) = \bigcap_{k \in I} \bigcap_{s=1}^{\infty} \bigcap_{n=1}^{\infty} E_n(I)$ and $G_n \rightarrow F$. The proof is complete. \square

The analogue of Theorem 4.1 follows immediately: if I is a non empty finite set or an infinitely countable set, then the set

$$T(I) = S(I) \cap iS(I) \quad (5.27)$$

is a dense and G_δ subset of $A(\Omega)$. This gives the non directional differentiability at every boundary point of both real and imaginary part of the function in every direction in $\ell^\infty(I) \setminus \{0\}$.

Remark 5.7 In [3] it is claimed that the set of functions $f \in A(D^{\mathbb{N}})$ for which for every $\theta \in \mathbb{R}^{\mathbb{N}}$ and every direction $v \in \mathbb{R}^{\mathbb{N}}$ with $\|v\|_{\infty} = 1$ and $\frac{1}{2} \leq |v_k| \leq 1$ for a fixed $k \in \mathbb{N}$ there exists $y \in \left(\theta - \frac{1}{n}v, \theta + \frac{1}{n}v\right)$ such that

$$|u(\theta) - u(y)| > n\|y - \theta\|_{\infty} \quad (5.28)$$

is open in $A(D^{\mathbb{N}})$. The authors' proof uses the wrong statement that the set $\{v \in \mathbb{R}^{\mathbb{N}} \text{ with } \|v\|_{\infty} = 1 \text{ and } \frac{1}{2} \leq |v_k| \leq 1\}$ for a fixed $k \in \mathbb{N}$ is compact in $\mathbb{R}^{\mathbb{N}}$ endowed with the product topology. However, their proof works just fine for every compact set of directions with non zero k -coordinate. According to our Theorem 5.6 their statement is correct.

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